# A CRITERION FOR MEMBERSHIP IN ARCHIMEDEAN SEMIRINGS

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Abstract. Let $A$ be a commutative ring and $T \subset A$ a weakly divisible archimedean semiring, i.e., $0 \in T$, $T + T \subset T$, $TT \subset T$, $\mathbb{Z} + T = A$ and $\frac{1}{r} \in T$ for some integer $r \geq 2$. The classical Real Representation Theorem says the following: If $a \in A$ satisfies $\varphi(a) > 0$ for all ring homomorphisms $\varphi : A \to \mathbb{R}$ with $\varphi(T) \subset [0, \infty)$, then $a \in T$.

The main drawback of this criterion for membership is that it is only sufficient but far from being necessary since $\varphi(a) > 0$ cannot be replaced by $\varphi(a) \geq 0$ without any further conditions. Initiated by work of Scheiderer, a lot of progress has previously been made in overcoming this drawback but only in the case where $T$ is a preorder, i.e., contains all squares of $A$.

A different approach enables us to prove a suitable extension of the Real Representation theorem for the general case. If (and, to the best of our knowledge, only if) $T$ is a preorder, our result can easily be derived by earlier work of Scheiderer, Kuhlmann, Marshall and Schwartz. In contrast to this earlier work, our proof does not use and therefore shows the classical theorem.

We illustrate the usefulness of our result by deriving a theorem of Handelman from it saying inter alia the following: If an odd power of a real polynomial in several variables has only nonnegative coefficients, then so do all sufficiently high powers.

## 1. Archimedean semirings

Throughout this article, $A$ denotes a commutative ring. The case where the unique ring homomorphism $\mathbb{Z} \to A$ (all rings have a unity and all ring homomorphisms preserve unities) is not an embedding is formally admitted but our results will be trivial in this case. So the reader might assume that $A$ contains $\mathbb{Z}$ as a subring. Whenever we postulate that $\frac{1}{r}$ lies in $A$ for some integer $r \geq 2$, we implicitly require that $r$ (that is $r \cdot 1$) is a unit of $A$ (i.e., invertible in $A$).

**Definition 1.** A set $T \subset A$ is called a *semiring* of $A$ if $0, 1 \in T$ and $T$ is closed under addition and multiplication, i.e., $T + T \subset T$ and $TT \subset T$. A semiring $T$ of $A$ is called a *preorder* of $A$ if it contains all the squares of $A$, i.e., $A^2 \subset T$. We call a semiring $T$ *archimedean* (with respect to $A$) if $\mathbb{Z} + T = A$. We call a semiring *weakly divisible* if there is some integer $r \geq 2$ with $\frac{1}{r} \in T$.

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Semirings in our sense (i.e., as subsets of rings) are often called *preprimes* (cf. [PD, Definition 5.4.1]). This goes back to Harrison who called these objects *infinite* preprimes (opposing them to his *finite* preprimes) which makes sense in a certain number theoretic context [Har]. However, without the adjective "infinite" and in a general context, this terminology is hermeneutic. We use the term "semiring" and hope that other authors will follow. Some authors require that a semiring ("preprime" in their terminology) does not contain $-1$. This leads to similar technical problems than it would to require all ideals of a ring to be proper.

This work provides a new criterion to prove membership in an archimedean semiring. We shortly explain the well-known basic ideas. Suppose $S$ is a compact space. In the ring $\mathcal{C}(S,\mathbb{R})$ of continuous functions on $S$, the nonnegative functions form an archimedean semiring $\mathcal{C}(S,[0,\infty))$ (which is even a preordering). This semiring is defined by a clear geometric property (namely being nonnegative on the space $S$). More interesting semirings $T$, however, are often defined in an algebraic way, for example by a set of generators. The question arises if one can nevertheless develop criteria for membership in $T$ of (as far as possible) geometric nature. The first step is to view the elements of $A$ as real-valued continuous functions on some topological space $S(T)$ (naturally associated to $A$ and $T$) such that the elements of $T$ are nonnegative on $S(T)$.

For any semiring $T \subset A$, we set

$$S(T) := \{\varphi \mid \varphi : A \to \mathbb{R} \text{ ring homomorphism}, \varphi(T) \subset [0,\infty)\} \subset \mathbb{R}^A$$

where the topology on $S(T)$ is induced by the product topology on $\mathbb{R}^A$, i.e., is the weakest topology making $S(T) \to \mathbb{R} : \varphi \mapsto \varphi(a)$ for all $a \in A$ continuous. If $T$ is archimedean, then $S(T)$ is compact (meaning quasi-compact and Hausdorff). This is clear because $S(T)$ equals the intersection (18)–(21) appearing in the proof of Theorem 10 below. We now have a ring homomorphism

$$A \to \mathcal{C}(S(T),\mathbb{R}) : a \mapsto (\varphi \mapsto \varphi(a))$$

sending all $a \in T$ to a function nonnegative on the whole of $S(T)$. When we write $a$, we will often mean the image under this map. In this sense, $\varphi(a) = a(x)$ for all $x := \varphi \in S(T)$.

Often, $S(T)$ takes on a very concrete form. Concerning our motivating example of the archimedean semiring $\mathcal{C}(S,[0,\infty)) \subset \mathcal{C}(S,\mathbb{R})$ where $S$ is a compact space, it follows from basic set topology that the canonical map $S \to S(\mathcal{C}(S,[0,\infty))) : x \mapsto (f \mapsto f(x))$ is a homeomorphism (cf. [GJ, 4.9(a)]) allowing us to write $S = S(T)$. This illustrates the naturality of the definition of $S(T)$. However, for this semiring $\mathcal{C}(S,[0,\infty))$ our membership criterion will be inferior to the self-evident one.

Our criterion will rather be interesting in the realm of polynomials. Throughout this article, we will consider polynomials in $n$ variables $\bar{X} := (X_1,\dots,X_n)$. The polynomial ring in these $n$ variables over a commutative ring $R$ will be denoted by $R[\bar{X}]$. For any set $P \subset \mathbb{R}[\bar{X}]$, we define

$$V(P) := \{x \in \mathbb{R}^n \mid p(x) = 0 \text{ for all } p \in P\} \subset \mathbb{R}^n.$$

Suppose that $A$ is finitely generated over a subring $R$. Then (up to isomorphism) $A = R[\bar{X}]/I$ for some number $n$ of variables and an ideal $I$ of $A$. If $R \subset \mathbb{R}$ and $[0,\infty) \cap R \subset T$, then every $\varphi \in S(T)$ is the identity on $R$ and it is easy to see that

$$S(T) = \{x \in V(I) \mid t(x) \geq 0 \text{ for all } t \in T\} \subset \mathbb{R}^n \tag{1}$$

via the homeomorphism

$$\varphi \mapsto (\varphi(X_1 + I), \dots, \varphi(X_n + I)).$$

In particular, if $A = \mathbb{R}[\bar{X}]$ and $T \subset A$ is finitely generated over $[0, \infty)$, say

$$T = \langle [0, \infty) \cup \{t_1, \dots, t_m\} \rangle$$

(we always write angular brackets for the generated semiring), then

$$S(T) = \{x \in \mathbb{R}^n \mid t_1(x) \geq 0, \dots, t_m(x) \geq 0\} \subset \mathbb{R}^n$$

is a so called basic closed semialgebraic set (cf. [PD, Theorem 2.4.1]). If in addition all $t_i$ are linear (i.e., of degree $\leq 1$), then $S(T)$ is a polyhedron. If this polyhedron $S(T)$ is compact (i.e., a polytope), then it follows from a well-known theorem on linear inequalities (cf. [PD, Theorem 5.4.5][H4]) and Proposition 2 below that $T$ is archimedean (and therefore also any semiring $T' \subset A$ containing $T$). As already mentioned, the converse is true in general: If $T$ is archimedean, then $S(T)$ is compact.

Without the linearity assumption on the $t_i \in \mathbb{R}[\bar{X}]$, Schmüdgen [Sch] showed that compactness of $S(T)$ implies (and therefore is equivalent) to the condition that the *preordering* $T' \supset T$ generated by $t_1, \dots, t_m$, i.e., the semiring

$$T' := \langle \mathbb{R}[\bar{X}]^2 \cup \{t_1, \dots, t_m\} \rangle \supset T$$

is archimedean.

Our criterion will extend the classical criterion which is Corollary 13 in this work. It is going back to Krivine, Stone, Kadison, Dubois and Becker. It used to be called Kadison-Dubois theorem but due to its (to some extent only recently revealed) complex history (see [PD, Section 5.6]) it is now often called Real Representation Theorem. It simply says that for a weakly divisible archimedean semiring $T \subset A$, all $a \in A$ with $a > 0$ on $S(T)$ lie in $T$.

Using what we said above, this implies for example Handelman's theorem that any polynomial positive on a polytope is a nonnegative linear combination of products of the linear polynomials defining the polytope [PD, Theorem 5.4.6][H4]. Also, it implies the corresponding weaker representation of polynomials positive on compact basic closed semialgebraic sets proved by Schmüdgen [PD, Theorem 5.2.9][Sch].

The main drawback of the Real Representation Theorem is that it is only a *sufficient* condition for membership because $f > 0$ on $S(T)$ cannot be replaced by $f \geq 0$ (for example, a nonzero polynomial having a zero in the interior of a polytope obviously never can allow Handelman's representation adressed above). The criterion we will prove in Section 2, Theorem 10 below, theoretically is necessary and sufficient. We say "theoretically" since it assumes the existence of a certain admissible identity and there is a trivial identity (namely $f = 1 \cdot f$) that is admissible if and only if $f \in T$. The Real Representation Theorem comes out as a special case since another trivial identity (namely $f = f \cdot 1$) is admissible if $f > 0$ on $S(T)$. Our criterion yields new insights when non-trivial admissible identities can be found. It is not of purely geometric but also of arithmetic nature.

A slightly less general criterion for membership in *preorderings* has recently been proved by Scheiderer [S3, Proposition 3.10]. It has been very successfully applied to partially extend Schmüdgen's representation from positive to nonnegative polynomials on compact basic closed semialgebraic sets. Section 3 is devoted to the question in how far our criterion goes beyond recent work of Scheiderer, Kuhlmann,

Marshall and Schwartz on preorderings. We will see that our Theorem 10 can easily be deduced from their work in the case of preorderings, but in the general case, a central lemma in their proof is no longer true (see Example 16). Our approach is therefore not only different from theirs but also applies to a significantly more general situation.

In Section 4, we apply our criterion to give for the first time a purely ring-theoretic proof of a nice theorem of Handelman saying inter alia the following: If some odd power of a real polynomial in several variables has only nonnegative coefficients, then so do all sufficiently high powers. See Theorem 22 and Corollary 23.

This example will nicely illustrate the following general principle: Even if a semiring $T \subset A$ is not archimedean, there is always a biggest subring $O_T(A) \subset A$ such that $T \cap O_T(A)$ is archimedean This follows from the important Proposition 2 below. So with some additional difficulties (namely determining $S(O_T(A))$), our membership criterion also gives information about non-archimedean semirings.

**Proposition 2.** *Let $T$ be a semiring of $A$. Then*

$$O_T(A) := \{a \in A \mid N \pm a \in T \textit{ for some } N \in \mathbb{N}\}$$

*is a subring of $A$, the* ring of $T$-bounded elements *of $A$. Moreover, $T$ is archimedean if and only if $O_T(A) = A$.*

*Proof.* Obviously, $0, 1 \in O_T(A)$ since $0 \pm 0 = 0 \in T$ and $1 \pm 1 \in \{0, 2\} \subset T$. It is immediate from the definition of $O_T(A)$ that $-O_T(A) \subset O_T(A)$. That $O_T(A)$ is closed under addition, follows easily from $T + T \subset T$. To see that it is closed under multiplication, use the two identities

$$3N^2 \mp ab = (N \mp a)(N + b) + N(N \pm a) + N(N - b)$$

and that $T$ is closed under multiplication and addition. We leave the second statement to the reader. □

Without going into details, we make some final remarks on the space $S(T)$. There is a larger topological space one could naturally associate to a semiring $T$ of a ring $A$, namely the subspace $\mathrm{Sper}_T(A)$ of the so-called real spectrum $\mathrm{Sper}(A)$ of $A$ consisting of all so-called orderings of the ring $A$ lying over $T$ (see, e.g., [PD, 4.1]). Since $S(T) \subset \mathrm{Sper}_T(A)$ via a canonical embedding, all our results will also be true for $\mathrm{Sper}_T(A)$. If $T$ is an archimedean semiring, then $S(T)$ equals $(\mathrm{Sper}_T(A))^{\max}$, the space of maximal orderings of $A$ lying above $T$. When $T$ is not archimedean, $\mathrm{Sper}_T(A)$ is certainly preferable to $S(T)$ (for example, $\mathrm{Sper}_T(A)$ is even then always quasi-compact). However, we feel that in the context of archimedean semirings we encounter here the usage of $\mathrm{Sper}_T(A)$ has only disadvantages. For example, unlike $S(T)$, $\mathrm{Sper}_T(A)$ can usually not be really identified with a concrete subset of $\mathbb{R}^n$. Confer also [S3, 2.3].

## 2. The membership criterion

We begin by introducing some notation. For $\alpha \in \mathbb{N}^n$, we write

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$

so that the monomial

$$\bar{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$$

has degree $|\alpha|$. For $x \in \mathbb{R}^n$, $\|x\|$ always denotes the 1-norm of $x$, i.e.,

$$\|x\| := |x_1| + \cdots + |x_n|.$$

Correspondingly,

$$B_r(x) := \{y \in \mathbb{R}^n \mid \|y - x\| < r\} \qquad (x \in \mathbb{R}^n, 0 < r \in \mathbb{R})$$

denotes the open ball around $x$ of radius $r$ with respect to the 1-norm and

$$\overline{B_r(x)} = \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$$

its closure. Like all norms, the 1-norm defines the usual topology on $\mathbb{R}^n$. The reason for our choice of this norm is that $\|\alpha\| = |\alpha|$ for $\alpha \in \mathbb{N}^n$. Despite this equality, we want to keep both notations since $|\alpha| = k$ will mean implicitly $\alpha \in \mathbb{N}^n$ (and that $\alpha$ plays the role of a tuple of exponents of a monomial $\bar{X}^\alpha$). We introduce the compact set

$$\begin{aligned}\Delta &:= (\overline{B_1(0)} \setminus B_1(0)) \cap [0, \infty)^n = V(\{X_1 + \cdots + X_n - 1\}) \cap [0, \infty)^n \\ &= \{x \in [0, \infty)^n \mid \|x\| = 1\} \subset \mathbb{R}^n.\end{aligned}$$

For a given set $P \subset \mathbb{R}[\bar{X}]$, we denote by $P^+$ its subset of all polynomials which have only nonnegative coefficients and by $P^*$ its subset of all homogeneous polynomials (i.e., all of whose nonzero monomials have the same degree).

Descartes had already the idea to relate the geometric properties of a real polynomial directly to combinatorial properties of the family of signs of its coefficients. His law of signs says that a real polynomial in one variable has not more positive real roots than it has sign changes in the sequence of its coefficients, and the difference is even [BPR, Theorem 2.34]. Given a sequence of signs, a good guess for the number of positive real roots of a corresponding polynomial would therefore perhaps be the number of these sign changes. Viro extended this naive rule of guessing the topological shape of the real zero set of a polynomial to the case of several variables. Given a pattern of signs, he can construct a corresponding polynomial whose real zero set has exactly the guessed shape. This is Viro's method for constructing real hypersurfaces with prescribed topology [Vir].

The starting point for the proof of our criterion is yet another idea in this vein going back to Pólya. Suppose $f \in \mathbb{R}[\bar{X}]^*$. Pólya relates the geometric behaviour of $f$ on the nonnegative orthant $[0, \infty)^n$ with the signs of the coefficients of a "refinement" of $f$. Due to homogeneity, $f$ can just as well be looked at on $\Delta$ instead of $[0, \infty)^n$. Multiplying $f$ by $X_1 + \cdots + X_n$ does not change $f$ on $\Delta$ but "refines" the pattern of signs of its coefficients. When we repeat this multiplication sufficiently often, it turns out that the obtained pattern reflects more and more the geometric sign behaviour of $f$ on $[0, \infty)^n$. The exact statement we will need is formulated in Lemma 3 below. Whereas previous works of the author [Sw1][Sw3][Sw4] (see Remark 14 below) required only Pólya's original theorem, we need this time really a more local version where we look at $f$ only on a closed subset $U$ of $\Delta$. Nevertheless, the proof goes exactly along the lines of Pólya (cf. [Pól][PR]). We include it for the convenience of the reader.

**Lemma 3.** *Suppose $f \in \mathbb{R}[\bar{X}]^*$ has degree $d$ and $U \subset \Delta$ is closed such that $f > 0$ on $U$. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $\alpha \in \mathbb{N}^n$ with ($k + d \neq 0$ and) $\frac{\alpha}{k+d} \in U$, the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ is positive.*

*Proof.* Write $f = \sum_{|\beta|=d} a_\beta \bar{X}^\beta$, $a_\beta \in \mathbb{R}$. We know that

$$(X_1 + \cdots + X_n)^k = \sum_{|\gamma|=k} \frac{k!}{\gamma_1! \cdots \gamma_n!} \bar{X}^\gamma$$

for $k \in \mathbb{N}$. Of course, if $\alpha \in \mathbb{N}^n$ with $\frac{\alpha}{k+d} \in U \subset \Delta$, then $|\alpha| = k + d$. Now for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = k+d$, the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ equals

$$\sum_{\substack{|\beta|=d, |\gamma|=k \\ \beta+\gamma=\alpha}} \frac{k!}{\gamma_1! \cdots \gamma_n!} a_\beta$$

$$= \sum_{\substack{|\beta|=d, |\gamma|=k \\ \beta+\gamma=\alpha}} \frac{k!}{(\alpha_1 - \beta_1)! \cdots (\alpha_n - \beta_n)!} a_\beta \qquad \text{(terms of the sum do not depend on } \gamma\text{)}$$

$$= \sum_{\substack{|\beta|=d \\ \beta\leq\alpha}} \frac{k!}{(\alpha_1 - \beta_1)! \cdots (\alpha_n - \beta_n)!} a_\beta \qquad (\beta \leq \alpha \text{ understood componentwise})$$

$$= \frac{k!(k+d)^d}{\alpha_1! \cdots \alpha_n!} \sum_{\substack{|\beta|=d \\ \beta\leq\alpha}} a_\beta \prod_{i=1}^n \frac{\alpha_i!}{(\alpha_i - \beta_i)!(k+d)^{\beta_i}} \qquad \text{(using } |\beta| = d\text{, provided } k+d \neq 0\text{)}$$

$$= \frac{k!(k+d)^d}{\alpha_1! \cdots \alpha_n!} \sum_{|\beta|=d} a_\beta \prod_{i=1}^n \left(\frac{\alpha_i}{k+d}\right)^{\beta_i}_{\frac{1}{k+d}} \qquad \left(\text{abbreviating } (a)^m_b := \prod_{i=0}^{m-1} (a - ib)\right).$$

Note that $(a)^m_0 = a^m$ to understand the idea behind the notation $(a)^m_b$ just introduced. Also note that the condition $\beta \leq \alpha$ has been dropped in the index of summation in the last expression. This is justified since all the corresponding additional terms in the sum are zero. Now we see that the coefficient in question equals (assuming $k + d \neq 0$) up to a positive factor

$$f_{\frac{1}{k+d}}\left(\frac{\alpha}{k+d}\right)$$

where we define

$$f_\varepsilon := \sum_{|\beta|=d} a_\beta (X_1)^{\beta_1}_\varepsilon \cdots (X_n)^{\beta_n}_\varepsilon \in \mathbb{R}[\bar{X}]$$

for all $\varepsilon \in [0, \infty)$. Obviously, $f_\varepsilon$ converges to $f_0 = f$ uniformly on $U$ when $\varepsilon \to 0$. Since $U$ is compact and $f > 0$ on $U$, there is $k_0 \in \mathbb{N}$ such that $f_{\frac{1}{k+d}} > 0$ on $U$ for all $k \geq k_0$, in particular

$$f_{\frac{1}{k+d}}\left(\frac{\alpha}{k+d}\right) > 0$$

whenever ($k + d \neq 0$ and) $\frac{\alpha}{k+d} \in U$. □

We draw from this Pólya's theorem as a corollary although we will never use it later. Note that Pólya's theorem follows as easily by taking independently of $x \in \Delta$ the same identity $f = f \cdot 1$ in condition (a) of Lemma 6 below.

**Corollary 4** (Pólya)**.** *Suppose $f \in \mathbb{R}[\bar{X}]^*$ and $f > 0$ on $\Delta$. Then*

$$(X_1 + \cdots + X_n)^k f \in \mathbb{R}[\bar{X}]^+$$

*for large $k \in \mathbb{N}$.*

*Proof.* Set $U = \Delta$ in Lemma 3. □

It is perhaps worth pointing out that Pólya's theorem is closely related to Bernstein polynomials. See [Far, Theorem 1.3] for a theorem on (generalized) Bernstein polynomials (in several variables) which is nothing else than a version of Pólya's theorem. Via this connection, Pólya's Theorem for the case of two variables (i.e., when $n = 2$) is connected to Descartes' law of signs mentioned above [BPR, Section 10.27]. For technical reasons, it is very convenient to have the following evident consequence of Lemma 3 available.

**Lemma 5.** *Suppose $f \in \mathbb{R}[\bar{X}]^*$ and $U \subset \Delta$ is closed such that $f > 0$ on $U$. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $0 \neq \alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in U$, the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ is nonnegative.*

*Proof.* Without loss of generality $f \neq 0$. Set $d := \deg f$ and choose $k_0$ like in the previous lemma. Let $k \geq k_0$ and $0 \neq \alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in U$. If $|\alpha| = k + d$, $\bar{X}^\alpha$ has a positive coefficient in $(X_1 + \cdots + X_n)^k f$ by the choice of $k_0$. If $|\alpha| \neq k + d$, the coefficient of $\bar{X}^\alpha$ in this same polynomial is zero since it is a homogeneous polynomial of degree $k + d$. □

The next lemma reminds already a bit of Theorem 10 below. But note that the $g_i$ and $h_i$ are allowed to depend on $x$. The idea is to apply Pólya's refinement process locally on the $g_i$ while the $h_i$ do not disturb too much. Note that we do no longer assume that $f$ is homogeneous. Also observe that the hypotheses imply $f \geq 0$ on $\Delta$.

**Lemma 6.** *Let $f \in \mathbb{R}[\bar{X}]$. Suppose that for every $x \in \Delta$ there are $m \in \mathbb{N}$, $g_1, \ldots, g_m \in \mathbb{R}[\bar{X}]^*$ and $h_1, \ldots, h_m \in \mathbb{R}[\bar{X}]^+$ such that*
*(a) $f = g_1 h_1 + \cdots + g_m h_m$ and*
*(b) $g_1(x) > 0, \ldots, g_m(x) > 0$.*
*Then there exists $k \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^k f \in \mathbb{R}[\bar{X}]^+$.*

*Proof.* Choose a family $(\varepsilon_x)_{x \in \Delta}$ of real numbers $\varepsilon_x > 0$ such that for every $x \in \Delta$, there are $m \in \mathbb{N}$, $g_1, \ldots, g_m \in \mathbb{R}[\bar{X}]^*$ and $h_1, \ldots, h_m \in \mathbb{R}[\bar{X}]^+$ satisfying (a) and not only (b) but even

$$g_i > 0 \text{ on } \overline{B_{2\varepsilon_x}(x)} \cap \Delta \qquad \text{for } i \in \{1, \ldots, m\}. \tag{2}$$

The family $(B_{\varepsilon_x}(x))_{x \in \Delta}$ is an open covering of $\Delta$. Since $\Delta$ is compact, there is a finite subcovering, i.e., a finite set $D \subset \Delta$ for which $\Delta \subset \bigcup_{x \in D} B_{\varepsilon_x}(x)$, in particular

$$\Delta = \bigcup_{x \in D} (\overline{B_{\varepsilon_x}(x)} \cap \Delta).$$

As $D$ is finite, it suffices to show for fixed $x \in D$, that there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $0 \neq \alpha \in \mathbb{N}^n$ with

$$\frac{\alpha}{|\alpha|} \in \overline{B_{\varepsilon_x}(x)}, \tag{3}$$

the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k f$ is nonnegative (note that $\frac{\alpha}{|\alpha|} \in \Delta$ is automatic).

Therefore fix $x \in D$. By choice of $\varepsilon_x$, we find $m \in \mathbb{N}$, $g_1, \ldots, g_m \in \mathbb{R}[\bar{X}]^*$ and $h_1, \ldots, h_m \in \mathbb{R}[\bar{X}]^+$ satisfying (a) and (2). For every $i \in \{1, \ldots, m\}$, the positivity

condition (2) enables us to apply Lemma 5 to $g_i$, yielding $k_i \in \mathbb{N}$ such that for all $k \geq k_i$ and all $0 \neq \beta \in \mathbb{N}^n$ with

$$\frac{\beta}{|\beta|} \in \overline{B_{2\varepsilon_x}(x)}, \tag{4}$$

the coefficient of $\bar{X}^\beta$ in $(X_1 + \cdots + X_n)^k g_i$ is nonnegative (use that $\frac{\beta}{|\beta|} \in \Delta$ is automatic). Choose moreover $1 \leq l \in \mathbb{N}$ so large that

$$\frac{2|\gamma|}{l} \leq \varepsilon_x \tag{5}$$

for all $\gamma \in \mathbb{N}^n$ for which the coefficient of $\bar{X}^\gamma$ in at least one of the polynomials $h_1, \dots, h_m$ does not vanish. Set

$$k_0 := \max\{k_1, \dots, k_n, l\}.$$

Let $k \geq k_0$ and suppose $0 \neq \alpha \in \mathbb{N}^n$ satisfies (3). Fix $i \in \{1, \dots, m\}$. By equation (a), it is enough to show that the coefficient of $\bar{X}^\alpha$ in $(X_1 + \cdots + X_n)^k g_i h_i$ is nonnegative. This coefficient is of course a sum of certain products of coefficients of $(X_1+\cdots+X_n)^k g_i$ and $h_i$. But all the concerned products are nonnegative. Indeed, consider $\beta, \gamma \in \mathbb{N}^n$ with $\beta + \gamma = \alpha$ (i.e., $\bar{X}^\beta \bar{X}^\gamma = \bar{X}^\alpha$) such that the corresponding coefficients of $\bar{X}^\beta$ in $(X_1 + \cdots + X_n)^k g_i$ and $\bar{X}^\gamma$ in $h_i$ do not vanish. The latter coefficient is positive since $h_i \in \mathbb{R}[\bar{X}]^+$. We show that the other one is positive, too. From degree consideration it is trivial that $|\beta| \geq k \geq k_0 \geq l \geq 1$ which implies together with the now satisfied condition (5)

$$\frac{2|\gamma|}{|\beta|} \leq \varepsilon_x. \tag{6}$$

We exploit this to verify condition (4) which is all we need since $k \geq k_0 \geq k_i$:

$$\begin{aligned}
\left\| \frac{\beta}{|\beta|} - x \right\| &\leq \left\| \frac{\beta}{|\beta|} - \frac{\alpha}{|\alpha|} \right\| + \underbrace{\left\| \frac{\alpha}{|\alpha|} - x \right\|}_{\leq \varepsilon_x \text{ by } (3)} \leq \varepsilon_x + \left\| \frac{|\alpha|\beta - |\beta|\alpha}{|\alpha||\beta|} \right\| \\
&= \varepsilon_x + \frac{1}{|\alpha||\beta|} \| \underbrace{|\alpha|(\gamma - \alpha) - \overbrace{|\gamma - \alpha|}^{=|\gamma|-|\alpha|} \alpha}_{=|\alpha|\gamma - |\alpha|\alpha - |\gamma|\alpha + |\alpha|\alpha} \| \\
&= \varepsilon_x + \frac{\| |\alpha|\gamma - |\gamma|\alpha \|}{|\alpha||\beta|} \leq \varepsilon_x + \frac{\| |\alpha|\gamma \| + \| |\gamma|\alpha \|}{|\alpha||\beta|} \\
&= \varepsilon_x + \frac{2|\alpha||\gamma|}{|\alpha||\beta|} = \varepsilon_x + \frac{2|\gamma|}{|\beta|} \overset{(6)}{\leq} 2\varepsilon_x
\end{aligned}$$

□

Now we deal with the case where the $g_i$ are no longer assumed to be homogeneous.

**Lemma 7.** *Let $f \in \mathbb{Z}[\bar{X}]$ such that for all $x \in \Delta$, there exist $m \in \mathbb{N}$, $g_1, \dots, g_m \in \mathbb{Z}[\bar{X}]$ and $h_1, \dots, h_m \in \mathbb{Z}[\bar{X}]^+$ such that*

*(a) $f = g_1 h_1 + \cdots + g_m h_m$ and*

*(b) $g_1(x) > 0, \dots, g_m(x) > 0$.*

*Then $f$ is modulo the principal ideal $\mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1)$ congruent to a polynomial without negative coefficients.*

*Proof.* For every $x \in \Delta$, choose $m_x \in \mathbb{N}$, $g_{x1}, \dots, g_{xm_x} \in \mathbb{Z}[\bar{X}]$ and $0 \neq h_{x1}, \dots, h_{xm_x} \in \mathbb{Z}[\bar{X}]^+$ according to (a) and (b). Setting

$$U_x := \{y \in \Delta \mid g_{x1}(y) > 0, \dots, g_{xm_x}(y) > 0\}, \tag{7}$$

we have $x \in U_x$ for $x \in \Delta$. Therefore $(U_x)_{x\in\Delta}$ is an open covering of the compact set $\Delta$ and possesses a finite subcovering, i.e., there is a finite set $D \subset \Delta$ such that

$$\Delta = \bigcup_{x\in D} U_x. \tag{8}$$

Choose an upper bound $d \in \mathbb{N}$ for the degrees of the (in each case $m_x$) terms appearing in the sums on the right hand sides of the equations (a) corresponding to the finitely many $x \in D$, i.e.,

$$d \geq \deg g_{xi} + \deg h_{xi} \qquad \text{for all } x \in D \text{ and } i \in \{1, \dots, m_x\}.$$

Fix for the moment such a pair $(x, i)$ and choose $d', d'' \in \mathbb{N}$ such that $d = d' + d''$, $d' \geq \deg g_{xi}$ and $d'' \geq \deg h_{xi}$. Write $g_{xi} = \sum_{k=0}^{d'} p_k$ and $h_{xi} = \sum_{k=0}^{d''} q_k$ where $p_k, q_k \in \mathbb{Z}[\bar{X}]$ are homogeneous of degree $k$ (if not zero). Set

$$g'_{xi} := \sum_{k=0}^{d'} (X_1 + \dots + X_n)^{d'-k} p_k \quad \text{and} \qquad h'_{xi} := \sum_{k=0}^{d''} (X_1 + \dots + X_n)^{d''-k} q_k.$$

Now $g'_{xi}$ and $h'_{xi}$ are homogeneous polynomials whose product is (homogeneous) of degree $d$ (if not zero). Then $g'_{xi} \equiv g_{xi}$ and $h'_{xi} \equiv h_{xi}$ modulo $\mathbb{Z}[\bar{X}](X_1+\dots+X_n-1)$, in particular, $g'_{xi}$ coincides with $g_{xi}$ on $\Delta$. Moreover, $h'_{xi}$ inherits the property of having no negative coefficients from $h_{xi}$. For every $x \in D$,

$$f'_x := g'_{x1}h'_{x1} + \dots + g'_{xm_x}h'_{xm_x} \in \mathbb{Z}[\bar{X}]^* \tag{9}$$

is homogeneous of degree $d$ (unless zero) and congruent to $f$ modulo $\mathbb{Z}[\bar{X}](X_1 + \dots + X_n - 1)$. For $x, y \in D$, $f'_x - f'_y$ is therefore homogeneous and at the same time a multiple of $X_1 + \dots + X_n - 1$. Hence actually $f'_x = f'_y$, i.e., there is $f' \in \mathbb{Z}[\bar{X}]$ such that $f' = f'_x$ for all $x \in D$ and $f' \equiv f$ modulo $\mathbb{Z}[\bar{X}](X_1 + \dots + X_n - 1)$.

We want to apply Lemma 6 to $f'$. The hypotheses are now rather easy to verify: Let $x \in \Delta$. By (8), we find $x \in D$ such that $x \in U_x$. Set $m := m_x$, $g_i := g'_{xi}$ and $h_i := h'_{xi}$ for $i \in \{1, \dots, m\}$. Then equation (9) becomes condition (a) in Lemma 6 (with $f'$ instead of $f$). To verify (b) of Lemma 6, use that $g_i = g'_{xi}$ equals $g_{xi}$ on $\Delta$ which is positive in $x \in U_x \subset \Delta$ by (7). By Lemma 6, we get therefore $k \in \mathbb{N}$ such that $(X_1 + \dots + X_n)^k f'$ has no negative coefficients. But this polynomial is congruent to $f'$ which is in turn congruent to $f$ modulo $\mathbb{Z}[\bar{X}](X_1+\dots+X_n-1)$. □

Compared to the lemma we just proved, the next statement has the big advantage that the principal ideal $\mathbb{Z}[\bar{X}](X_1+\dots+X_n-1)$ can be replaced by any larger ideal. On the other hand, the $h_i$ are no longer allowed to depend on $x$. This disadvantage is made more tolerable by the fact that only those $x$ have to be considered where $f$ vanishes. In the previous (but not in the next) lemma this fact is implicitly obvious since $f = f \cdot 1$ is an admissible identity at the points where $f$ is positive.

**Lemma 8.** *Let $I$ be an ideal of $\mathbb{Z}[\bar{X}]$ such that $X_1 + \dots + X_n - 1 \in I$. Suppose $m \in \mathbb{N}$, $f \in \mathbb{Z}[\bar{X}]$ and $h_1, \dots, h_m \in \mathbb{Z}[\bar{X}]^+$ such that*
*(a) $f \geq 0$ on $V(I) \cap [0,\infty)^n$ and*
*(b) for all $x \in V(I \cup \{f\}) \cap [0,\infty)^n$, there exist $g_1, \dots, g_m \in \mathbb{Z}[\bar{X}]$ such that*

*(i)* $f = g_1h_1 + \cdots + g_mh_m$ *and*
*(ii)* $g_1(x) > 0, \ldots, g_m(x) > 0$.

*Then* $f$ *is modulo* $I$ *congruent to a polynomial without negative coefficients.*

*Proof.* Set $U := \{x \in \Delta \mid f(x) > 0\}$ and introduce the set $W \subset \Delta$ of all $x \in \Delta$ for which there are $g_1, \ldots, g_m$ fulfilling (i) and (ii). The sets $U$ and $W$ are open in $\Delta$ and

$$V(I) \cap [0,\infty)^n \subset U \cup W \tag{10}$$

by (a) and (b). By Hilbert's Basis Theorem, every ideal of $\mathbb{Z}[\bar{X}]$ is finitely generated. In particular, we find $s \in \mathbb{N}$ and $p_1, \ldots, p_s \in \mathbb{Z}[\bar{X}]$ such that

$$I = \mathbb{Z}[\bar{X}]p_1 + \cdots + \mathbb{Z}[\bar{X}]p_s + \mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1).$$

Setting $p := \sum_{i=1}^s p_i^2 \in I$, we have $p \in I$, $p \geq 0$ on $\mathbb{R}^n$ and

$$p > \varepsilon \text{ on } \Delta \setminus (U \cup W) \qquad \text{for some } \varepsilon > 0. \tag{11}$$

The latter follows from $p > 0$ on $\Delta \setminus V(I)$, (10) and the compactness of $\Delta \setminus (U \cup W)$.

Now we distinguish two cases. First case: $W = \emptyset$. From (11) and the boundedness of $f$ on the compact set $\Delta \setminus U$, we get $k \in \mathbb{N}$ such that $f' := f + kp > 0$ on $\Delta \setminus U$. On the other hand, $f' = f + kp \geq f > 0$ on $U$. Altogether we get $f' > 0$ on $\Delta$. Now we can clearly apply Lemma 7 to $f'$. In fact, for every $x \in \Delta$, $f' = f' \cdot 1$ serves as an identity as required in (a) of that lemma. Hence that lemma yields that $f'$ is congruent to a polynomial without negative coefficients modulo $\mathbb{Z}[\bar{X}](X_1 + \cdots + X_n - 1) \subset I$. But $f \equiv f + kp = f'$ modulo $I$.

Second case: $W \neq \emptyset$. All we really use from $W \neq \emptyset$ is that $f \in \mathbb{Z}[\bar{X}]h_1 + \cdots + \mathbb{Z}[\bar{X}]h_m$ by (i), i.e., we find $q_1, \ldots, q_m \in \mathbb{Z}[\bar{X}]$ such that

$$f = q_1h_1 + \cdots + q_mh_m. \tag{12}$$

From (11) and the boundedness of $q_1, \ldots, q_m$ on the compact set $\Delta \setminus (U \cup W)$, it follows that we can choose $k \in \mathbb{N}$ such that

$$g_i^{(0)} := q_i + kp > 0 \qquad \text{on } \Delta \setminus (U \cup W) \text{ for all } i \in \{1, \ldots, m\}. \tag{13}$$

We will apply Lemma 7 to

$$f' := g_1^{(0)}h_1 + \cdots + g_m^{(0)}h_m. \tag{14}$$

Note that

$$f' \overset{(13)}{=} \underbrace{q_1h_1 + \cdots + q_mh_m}_{= f \text{ by } (12)} + kp(\underbrace{h_1 + \cdots + h_m}_{\geq 0 \text{ on } [0,\infty)^n}) \geq f \qquad \text{on } [0,\infty)^n. \tag{15}$$

To check its applicability, let $x \in \Delta$. We consider three different subcases:

First, consider the case where $x \in U$. Then $f'(x) \geq f(x) > 0$ and

$$f' = f' \cdot 1 \tag{16}$$

is an identity as demanded in (a) of Lemma 7.

Second, suppose $x \in W$. By definition of $W$, we can choose $g_1, \ldots, g_m \in \mathbb{Z}[\bar{X}]$ satisfying $(i)$ and $(ii)$. Set $g_i' := g_i + kp$ for $i \in \{1, \ldots, m\}$. Then

$$\begin{aligned} f' &\overset{(15)}{=} f + kp(h_1 + \cdots + h_m) \\ &\overset{(i)}{=} g_1h_1 + \cdots + g_mh_m + kp(h_1 + \cdots + h_m) = g_1'h_1 + \cdots + g_m'h_m \end{aligned} \tag{17}$$

serves as a relation as required in (a) of Lemma 7. Note that

$$g_i'(x) = g_i(x) + kp(x) \geq g_i(x) \overset{(ii)}{>} 0 \qquad \text{for } i \in \{1, \dots, m\}.$$

Third and last, for all $x \in \Delta \setminus (U \cup W)$, (13) allows us to use one and the same equation for (a) of Lemma 7, namely (14).

All in all, Lemma 7 applies now to $f'$, i.e., $f'$ is congruent to a polynomial without nonnegative coefficients modulo $\mathbb{Z}[\bar{X}](X_1 + \dots + X_n - 1) \subset I$. But $f \equiv f + kp = f'$ modulo $I$. □

*Remark* 9. In Lemma 7, the $h_i$ are permitted to depend on $x$. In the proof of Lemma 8, we do not exploit this too much. Indeed, the three used identities (16), (17) and (14) are based on the same $h_i$ except (16) which is a trivial identity.

For any element $a \in A$, we set

$$S_{a=0}(T) := \{x \in S(T) \mid a(x) = 0\}.$$

Now we attack the main theorem. Note that its hypotheses imply that all $t_i$ vanish on $S_{a=0}(T)$.

**Theorem 10.** *Let $T$ be a weakly divisible archimedean semiring of $A$ and $a \in A$. Suppose $a \geq 0$ on $S(T)$ and there is an identity $a = b_1 t_1 + \dots + b_m t_m$ with $b_i \in A$, $t_i \in T$ such that $b_i > 0$ on $S_{a=0}(T)$ for all $i \in \{1, \dots, m\}$. Then $a \in T$.*

*Proof.* If the ring homomorphism $\mathbb{Z} \to A$ is not injective, then $-1 \in T$ whence $T = \mathbb{Z} + T = A$. Therefore we assume from now on that $A$ contains $\mathbb{Z}[\frac{1}{r}]$ as a subring and $\frac{1}{r} \in T$ for some integer $r \geq 2$. Because $T$ is archimedean, we find for every $c \in A$ some $N_c \in \mathbb{N}$ with $N_c \pm c \in T$. The topological space

$$S := \prod_{c \in A} [-N_c, N_c]$$

is compact by Tychonoff's theorem. From the hypotheses of the theorem, it follows that a certain intersection of closed subsets of $S$ is empty:

$$\bigcap_{c,d \in A} \{\varphi \in S \mid \varphi(c) + \varphi(d) - \varphi(c+d) = 0\} \quad \cap \tag{18}$$

$$\bigcap_{c,d \in A} \{\varphi \in S \mid \varphi(c)\varphi(d) - \varphi(cd) = 0\} \quad \cap \tag{19}$$

$$\{\varphi \in S \mid \varphi(1) = 1\} \quad \cap \tag{20}$$

$$\bigcap_{t \in T} \{\varphi \in S \mid \varphi(t) \geq 0\} \quad \cap \tag{21}$$

$$\{\varphi \in S \mid \varphi(a) \leq 0\} \quad \cap \tag{22}$$

$$\bigcup_{i=1}^{m} \{\varphi \in S \mid \varphi(b_i) \leq 0\} \quad = \emptyset \tag{23}$$

All sets appearing as subexpressions of (18)–(23) are closed. This is easy to see: Use that $\{0\}, \{1\}, [0, \infty), (-\infty, 0]$ are closed subsets of $\mathbb{R}$, that the projection maps $S \to \mathbb{R} : \varphi \mapsto \varphi(c)$ $(c \in A)$ are continuous (the characteristic property of the product topology), that $+, -, \cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and that finite unions and arbitrary intersections of closed sets are again closed.

Since $S$ is compact, some finite subintersection of (18)–(23) is already empty. In particular, (18)–(23) is already empty if the intersection in (18) and in (19) runs only over certain finitely many $c, d \in A$. Let $\bar{y} = (y_1, \dots, y_n)$ be the collection of $\frac{1}{r}$, all $b_i$, $t_i$ and these $c, d$. We claim that all hypotheses of the theorem remain valid for $(\mathbb{Z}[\bar{y}], T \cap \mathbb{Z}[\bar{y}])$ instead of $(A, T)$.

Indeed, first of all, $T \cap \mathbb{Z}[\bar{y}]$ inherits the property of being a weakly divisible archimedean semiring from $T$. Second, the identity from the hypotheses remains trivially satisfied (do not forget that $\mathbb{Z}[\bar{y}]$ contains $a$ since it contains all $b_i, t_i$ and it is a ring). Third and last, it remains to check that the geometric hypotheses stay valid. To this purpose, let $\varphi : \mathbb{Z}[\bar{y}] \to \mathbb{R}$ be a ring homomorphism with $\varphi(T \cap \mathbb{Z}[\bar{y}]) \subset [0, \infty)$. Extend $\varphi$ to a map

$$\psi : A \to \mathbb{R} : c \mapsto \begin{cases} 0 & \text{if } c \notin \mathbb{Z}[\bar{y}], \\ \varphi(c) & \text{if } c \in \mathbb{Z}[\bar{y}]. \end{cases}$$

We have $\psi \in S$: If $c \in A \setminus \mathbb{Z}[\bar{y}]$, then $\psi(c) = 0 \in [-N_c, N_c]$. If $c \in \mathbb{Z}[\bar{y}]$, then $N_c \pm c \in T \cap \mathbb{Z}[\bar{y}]$ by choice of $N_c$, whence

$$N_c \pm \psi(c) = N_c \pm \varphi(c) = \varphi(N_c \pm c) \in \varphi(T \cap \mathbb{Z}[\bar{y}]) \subset [0, \infty) \tag{24}$$

showing also in this case $\psi(c) \in [-N_c, N_c]$. Since $\varphi$ is a ring homomorphism, its extension $\psi$ satisfies the corresponding homomorphy conditions on $\bar{y}$. Therefore $\psi \in S$ lies in the finite subintersection of (18)–(19) that led us above to the choice of $\bar{y}$. It is even easier to see that $\psi$ also lies in the intersection (20)–(21). Because $\psi$ cannot lie in the empty set, $\psi$ cannot lie in both (22) and (23). Hence $\varphi(a) = \psi(a) > 0$ or, for all $i$, $\varphi(b_i) = \psi(b_i) > 0$. In the latter case $\varphi(a) = \varphi(b_1)\varphi(t_1) + \dots + \varphi(b_m)\varphi(t_m) \geq 0$. Altogether, this shows $a \geq 0$ on $S(T \cap \mathbb{Z}[\bar{y}])$ and $b_i > 0$ on $S_{a=0}(T \cap \mathbb{Z}[\bar{y}])$ for all $i$. In other words, the hypotheses of the theorem are valid for $(\mathbb{Z}[\bar{y}], T \cap \mathbb{Z}[\bar{y}])$ instead of $(A, T)$. So we can assume from now on that

$$A = \mathbb{Z}[\bar{y}], \tag{25}$$

i.e., that $A$ is finitely generated as a ring. We still let $r \geq 2$ be an integer such that $\frac{1}{r} \in T$. We assume that

$$y_i \in T \qquad \text{for all } i \in \{1, \dots, n\}. \tag{26}$$

This is justified by the fact that $T$ is archimedean since there is $N \in \mathbb{N}$ with $N + y_i \in T$ and we may replace $y_i$ by $N + y_i$ (this does not affect (25)). Moreover, the assumption

$$y_1 + \dots + y_n = 1 \tag{27}$$

is without loss of generality: We can extend $\bar{y}$ by $N - (y_1 + \dots + y_n)$ for any $N \in \mathbb{N}$ without harming (25). If we choose $N \in \mathbb{N}$ so large that $N - (y_1 + \dots + y_n) \in T$, then (26) remains valid at the same time. Choosing this $N$ even more carefully, namely as a power $r$, establishes (27) with a power of $r$ instead of 1 on the right hand side. Finally, divide each $y_i$ by this power of $r$ (cf. (25)).

Now consider the ring epimorphism $\mathbb{Z}[\bar{X}] \to \mathbb{Z}[\bar{y}]$ mapping $X_i$ to $y_i$ for every $i \in \{1, \dots, n\}$. Calling its kernel $I$, it induces a ring isomorphism $\mathbb{Z}[\bar{X}]/I \to A$ mapping $X_i + I$ to $y_i$. Without loss of generality, we may assume

$$A = \mathbb{Z}[\bar{X}]/I \qquad \text{and} \qquad y_i = X_i + I \text{ for } i \in \{1, \dots, n\}. \tag{28}$$

As explained in Section 1, we then have the concrete description (1) of $S(T)$, i.e.,

$$S(T) = \{x \in V(I) \mid t(x) \geq 0 \text{ for all } t \in T\} \subset \mathbb{R}^n. \tag{29}$$

The geometric part of the hypotheses of our theorem implies

$$\begin{aligned}
&(30) \qquad && \bigcap_{t\in T}\{x \in \Delta \cap V(I) \mid t(x) \geq 0\} && \cap \\
&(31) \qquad && \{x \in \Delta \cap V(I) \mid a(x) \leq 0\} && \cap \\
&(32) \qquad && \bigcup_{i=1}^{m}\{x \in \Delta \cap V(I) \mid b_i(x) \leq 0\} && = \emptyset.
\end{aligned}$$

This is analogous to the above intersection (18)–(23): $V(I)$ plays the role of subintersection (18)–(20) and $\Delta$ plays the role of $S$. Exactly as we would even get an empty intersection in (18)–(23) above with $S$ replaced by $\mathbb{R}^A$, we could replace here $\Delta$ by $\mathbb{R}^n$. But in order to have an intersection of closed subsets of a *compact* space, we have defined all sets as subsets of $S$ above and define them as subsets of $\Delta$ here. The fact that everything now happens in $\mathbb{R}^n$ instead of $\mathbb{R}^A$ is important. It will allow us to pass over to a finitely generated semiring $T' \subset T$. See Remark 12 below.

As already pointed out, (30)–(32) is an empty intersection of closed sets in the compact space $\Delta \cap V(I)$. Hence it has a finite empty subintersection. In particular, it is already empty if the intersection in (30) runs only over finitely many (instead of all) $t \in T$. Let $T' \subset A$ be the semiring generated by these finitely many $t$ and $\frac{1}{r}, y_1, \ldots, y_n, t_1, \ldots, t_m$.

The semiring $T'$ still is (weakly divisible and) archimedean. According to Proposition 2 and (25), this can be verified by checking $y_1, \ldots, y_n \in O_{T'}(A)$. But this is immediate from $1 + y_i \in 1 + T' \subset T'$ and

$$1 - y_i \overset{(27)}{=} \sum_{j \neq i} y_j \in T'$$

for $i \in \{1, \ldots, n\}$.

Next, we claim that $a \geq 0$ on $S(T')$ and $b_i > 0$ on $S_{a=0}(T')$. So let $x \in S(T')$. With respect to a description of $S(T')$ analogous to (29), we have of course $x \in V(I)$. From (26) with $T'$ instead of $T$ and (27), we obtain $x \in \Delta$. Therefore $x$ is contained in the finite subintersection of (30) which led to the choice of $T'$. So it cannot be contained in intersection (31)–(32). So, if $a(x) \leq 0$, then $b_i(x) > 0$ for all $i$, whence $a(x) \geq 0$ (so actually $a(x) = 0$) by the identity from the hypotheses (recall that $t_1, \ldots, t_m \in T'$).

Now, we see that the hypotheses of the theorem remain satisfied with $T$ substituted by $T'$. As $T' \subset T$, it is consequently enough to show the theorem for $T'$ instead of $T$. The advantage is that $T'$ is finitely generated as a semiring. For ease of notation, we work again with $T$ instead of $T'$ but can assume from now on that $T$ is a finitely generated semiring. But then we see that we could have chosen $y_1, \ldots, y_n$ fulfilling (25) in such a way that they generate $T$. Let us assume henceforth that we did so. Then it follows from (28) that

$$T = \{p + I \mid p \in \mathbb{Z}[\bar{X}]^+\}. \tag{33}$$

We see from this that

$$S(T) = V(I) \cap [0, \infty)^n. \tag{34}$$

Choose $g_1, \dots, g_m \in \mathbb{Z}[\bar{X}]$ and $h_1, \dots, h_m \in \mathbb{Z}[\bar{X}]^+$ such that $b_i = g_i + I$, $t_i = h_i + I$ for all $i$ (use (33)). Now set

$$f := g_1 h_1 + \dots + g_m h_m \in \mathbb{Z}[\bar{X}] \tag{35}$$

which is nothing else than condition (i) in Lemma 8. The remaining hypotheses of Lemma 8 are now provided by (27), (28) and (34). That lemma yields that $f$ is congruent to a polynomial without negative coefficients modulo $I$. By (33), this means that $a = f + I \in T$. □

Together with Remark 9, the next remark will tell us that the intermediate results in this section have not been exploited to their full extent. This gives hope that the just proved theorem can still be improved at least in certain special situations.

*Remark* 11. In condition (b) of Lemma 8, the $g_i$ are allowed to depend on $x$. When we apply this lemma in Theorem 10, we do not make use of this. One might suspect that we could therefore formulate Theorem 10 in greater generality, namely that we could permit the $b_i$ to vary locally. This seems to be a false conclusion: The problem seems to be that the identity in the hypotheses of Theorem 10 is an identity in the ring $A$ whereas (i) in Lemma 8 is really on the level of polynomials. If the $b_i$ depended on $x \in S(T)$, then also the $g_i$ in equation (35) and we could not keep the left hand side of (35) constant.

*Remark* 12. One is tempted to think that, in the preceding proof, the passage from $T$ to the finitely generated semiring $T' \subset T$ would better have been carried out already when choosing the finite empty subintersection of (18)–(23). Though we could indeed have let run intersection (21) only over finitely many $t$ (analogously to intersections (18) and (19)), we then would not have known how to show (24) which was absolutely necessary to show $\psi \in S$.

**Corollary 13** (Real Representation Theorem)**.** *Let $T$ be a weakly divisible archimedean semiring of $A$. Suppose that $a \in A$ satisfies $a > 0$ on $S(T)$. Then $a \in T$.*

*Proof.* Use $a = a \cdot 1$ as the required identity in the previous theorem. □

*Remark* 14. It is instructive to look how this section could be thinned out when one is content with proving (rather than extending) the just stated Real Representation Theorem. The whole proof then collapses into what is essentially already contained in the author's earlier work [Sw1] (see also [Sw3]). In the same way than [Sw1] therefore can be read as a proof of the Real Representation Theorem, the author's approach [Sw4, Section 2] to Putinar's Theorem [Put][PD, Theorem 5.3.8] via Pólya's theorem (Corollary 4 above) can be read as a proof of Jacobi's variant of the Real Representation Theorem [Jac][PD, Theorem 5.3.6]. Jacobi's variant says that Theorem 13 holds for *quadratic modules* au lieu of semirings where $T \subset A$ is called a quadratic module if $0, 1 \in T$, $T + T \subset T$ and $A^2 T \subset T$. Scheiderer recently extended also this membership criterion of Jacobi from positive to certain nonnegative elements [S2, Proposition 1.4] (see also [M, p. 2, footnote 1]). But the author's mentioned approach via Pólya's theorem to Jacobi's criterion seems not to be extendable to this recent result of Scheiderer.

## 3. Alternative proof for preorders

In this section, we demonstrate that Theorem 10 can easily be deduced from recent work of Scheiderer, Kuhlmann, Marshall and Schwartz but only in the case

where $T$ is a preorder. The following key lemma and its proof is essentially [KMS, Corollary 2.2].

**Lemma 15** (Kuhlmann, Marshall, Schwartz). *Let $T$ be an archimedean preorder of $A$. Suppose $1 \in Aa + Ab$, $a, b \geq 0$ on $S(T)$ and $ab \in T$. Then $a, b \in T$.*

*Proof.* By our hypothesis and [KMS, Lemma 2.1] (see also [S3, Proposition 2.7] or [M, Lemma 3.2] for a natural generalization of this not needed here), we have $s, t \in A$ such that $1 = sa + tb$ and $s, t > 0$ on $S(T)$. By the classical Real Representation Theorem 13, we have $s, t \in T$. Now $a = sa^2 + tab \in T$ (here we use that $A^2 \subset T$). Symmetrically, we have of course $b \in T$. □

The next example shows that this key lemma does no longer hold in the general situation where $T$ is only assumed to be a semiring instead of a preorder.

*Example* 16. Let $A := \mathbb{R}[X]$ and $T \subset A$ be the semiring generated by $[0, \infty)$ and the three polynomials $1 \pm X$ and $X^2 + X^4$. The elements of $T$ are the nonnegative linear combinations of products of these polynomials. By Proposition 2, $T$ is clearly archimedean. Setting $a := X^2$ and $b := 1 + X^2$, we clearly have $1 \in Aa + Ab$ and $ab \in T$. Being sums of squares, $a$ and $b$ are of course nonnegative on $S(T)$. We claim that $a \notin T$. Otherwise, we would have an identity

$$X^2 = \sum_{\alpha \in \mathbb{N}^3} \lambda_\alpha (1 + X)^{\alpha_1} (1 - X)^{\alpha_2} (X^2 + X^4)^{\alpha_3} \qquad (\lambda_\alpha \geq 0).$$

Evaluating at 0, we would get that the sum over all $\lambda_\alpha$ with $\alpha_3 = 0$ is 0. But then, those $\lambda_\alpha$ would have to equal zero since they are nonnegative. As a consequence, $X^2 + X^4$ would divide $X^2$ which is absurd.

The idea for the next proof is from Corollaries 2.3 and 2.4 in [KMS].

*Alternative proof of Theorem 10 in case $A^2 \subset T$.* The set $T' := T - a^2 T \subset A$ is an archimedean preorder and we have $S_{a=0}(T) = S(T')$. By hypothesis, we have therefore $b_i > 0$ on $S(T')$ for all $i$. From the classical Real Representation Theorem 13, we obtain $b_i \in T'$ for all $i$. Regarding the identity from the hypotheses, this entails $a \in T'$, i.e., $a(1 + at) \in T$ for some $t \in T$. By Lemma 15, therefore $a \in T$. □

Even if Lemma 15 were true for semirings instead of preorders (which is not the case), this alternative proof would break down. We would have to replace the preordering $T'$ generated by $T$ and $-a^2$ by the semiring $T - a^2 T + a^4 T - a^6 T + \ldots$ generated by $T$ and $-a^2$. But then we would get only that

$$a(1 + at_1 - a^3 t_3 + a^5 t_5 - a^7 t_7 + \ldots) \in T \qquad \text{for some } t_1, t_3, \ldots \in T$$

instead of $a(1 + at) \in T$ for some $t \in T$. The negative signs appearing in the second factor of this product now prevent us from applying Lemma 15.

## 4. Handelman's Theorem on powers of polynomials

In this section, we show that Theorem 10 can be used to give a new proof of a nice theorem of Handelman on powers of polynomials. See Theorem 22 and Corollary 23 below. The original proof in [H5] relies on some nontrivial facts from a whole theory of a certain class of partially ordered abelian groups which is to a large extent due to Handelman. Some of the used facts would not make sense in our ring-theoretic setting, e.g., [H3, Proposition I.2(c)]. We have decided to expose

the whole material we need though a big part of it can be found in less algebraic terminology in Handelman's original work [H1][H5] and in another new exposition of part of Handelman's theory [AT]. This is not only because we want to keep this article self-contained but also because we want to take on a new valuation theoretic viewpoint. We will however only use the most basic facts and notions from valuation theory as they can be found, for example, in the appendix of [PD].

At first glance, it seems that our theorem is not suitable to prove Theorem 22. Indeed, $\mathbb{R}[\bar{X}]^+$ is not an archimedean semiring of $\mathbb{R}[\bar{X}]$. However, for a semiring $T$ of a ring $A$, $T \cap O_T(A)$ is an archimedean semiring of the ring of $T$-bounded elements $O_T(A) \subset A$ (cf. Lemma 2). Still, this does not seem to help since $O_{\mathbb{R}[\bar{X}]^+}(\mathbb{R}[\bar{X}]) = \mathbb{R}$. When a ring of bounded elements is too small, it is often a good idea to localize it by a fixed element, i.e., to build a new ring where division by this element is allowed (see, e.g., [Sw2, Theorem 5.1] or [PV]). Following Handelman (see, e.g., [H3, p. 61]), we will localize by a fixed $0 \neq g \in \mathbb{R}[\bar{X}]^+$. Hence we consider the ring

$$\mathbb{R}[\bar{X}]_g := \mathbb{R}\left[\bar{X}, \frac{1}{g}\right] = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}], k \in \mathbb{N}\right\} \subset \mathbb{R}(\bar{X})$$

($\mathbb{R}(\bar{X})$ denoting the quotient field of $\mathbb{R}[\bar{X}]$) together with the semiring

$$T_g := \left\langle T \cup \left\{\frac{1}{g}\right\}\right\rangle = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}]^+, k \in \mathbb{N}\right\} \subset \mathbb{R}[\bar{X}]_g$$

(we write angular brackets for the generated semiring). For a polynomial $p \in \mathbb{R}[\bar{X}]$, we denote by $\operatorname{Log}(p) \subseteq \mathbb{N}^n$ the set of all $\alpha \in \mathbb{N}^n$ for which the coefficient of $\bar{X}^\alpha$ in $p$ does not vanish. Its convex hull $\operatorname{New}(p) \subset \mathbb{R}^n$ is called the *Newton polytope* of $p$. It is easy to see that

$$\operatorname{Log}(pq) \subset \operatorname{Log}(p) + \operatorname{Log}(q) \qquad \text{for all } p, q \in \mathbb{R}[\bar{X}], \tag{36}$$

$$\operatorname{Log}(pq) = \operatorname{Log}(p) + \operatorname{Log}(q) \qquad \text{for all } p, q \in \mathbb{R}[\bar{X}]^+ \text{ and} \tag{37}$$

$$\operatorname{New}(pq) = \operatorname{New}(p) + \operatorname{New}(q) \qquad \text{for all } p, q \in \mathbb{R}[\bar{X}]. \tag{38}$$

These basic facts will frequently be used in the sequel, most often tacitly. We now determine the ring of $T_g$-bounded elements $A(g)$ and its (by Proposition 2) archimedean semiring $T(g) := T_g \cap A_g$:

$$A(g) := O_{T_g}(A_g) = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}], k \in \mathbb{N}, \operatorname{Log}(f) \subset \operatorname{Log}(g^k)\right\} \subset A_g \tag{39}$$

$$T(g) := T_g \cap A(g) = \left\{\frac{f}{g^k} \mid f \in \mathbb{R}[\bar{X}]^+, k \in \mathbb{N}, \operatorname{Log}(f) \subset \operatorname{Log}(g^k)\right\} \subset A(g) \tag{40}$$

The inclusions from right to left are trivial whereas the inclusion from left to right in (39) uses (36) and the one in (40) uses (36) and (37). Using (36), the following becomes clear quickly:

$$A(g) = \mathbb{R}\left[\frac{\bar{X}^\alpha}{g} \mid \alpha \in \operatorname{Log}(g)\right] \tag{41}$$

$$T(g) = \left\langle [0, \infty) \cup \left\{\frac{\bar{X}^\alpha}{g} \mid \alpha \in \operatorname{Log}(g)\right\}\right\rangle \tag{42}$$

Fix an arbitrary $w \in \mathbb{R}^n$. There is exactly one valuation $v_w : \mathbb{R}(\bar{X}) \to \mathbb{R} \cup \{\infty\}$ satisfying

$$v_w(p) = -\max\{\langle w, \alpha\rangle \mid \alpha \in \mathrm{Log}(p)\} \qquad (0 \neq p \in \mathbb{R}[\bar{X}]). \tag{43}$$

This is easy to show by noting that $\mathrm{Log}(p)$ can be replaced by $\mathrm{New}(p)$ in (43) and using (38). Here and elsewhere $\langle w, \alpha\rangle$ denotes the usual scalar product of $w$ and $\alpha$. We define the *$w$-initial part* $\mathrm{in}_w(p) \in \mathbb{R}[\bar{X}]$ of a polynomial $p \in \mathbb{R}[\bar{X}]$ as the sum of those monomials appearing in $p$ belonging to an exponent tuple $\alpha \in \mathbb{N}^n$ for which $\langle w, \alpha\rangle$ gets maximal (i.e., equals $-v_w(p)$). The following is easy to check:

$$\mathrm{in}_w(p)(x) = \lim_{t\to\infty} e^{tv_w(p)} p(e^{tw_1}x_1, \dots, e^{tw_n}x_n) \qquad (0 \neq p \in \mathbb{R}[\bar{X}], x \in \mathbb{R}^n) \tag{44}$$

$$\mathrm{in}_w(pq) = \mathrm{in}_w(p)\,\mathrm{in}_w(q) \qquad (p, q \in \mathbb{R}[\bar{X}]) \tag{45}$$

Let $\mathcal{O}_w$ denote the valuation ring belonging to $v_w$ and $\mathfrak{m}_w$ its maximal ideal. It is an easy exercise to show that a ring homomorphism $\lambda_w : \mathcal{O}_w \to \mathbb{R}(\bar{X})$ having kernel $\mathfrak{m}_w$ is defined by

$$\lambda_w\left(\frac{p}{q}\right) := \begin{cases} 0 & \text{if } v_w(p) > v_w(q) \\ \frac{\mathrm{in}_w(p)}{\mathrm{in}_w(q)} & \text{if } v_w(p) = v_w(q) \end{cases} \qquad (p, q \in \mathbb{R}[\bar{X}], q \neq 0), \tag{46}$$

i.e., $\lambda_w$ is a place belonging to $v_w$.

We now give a concrete description of $S(T(g))$ using the notions just defined. This result is from Handelman [H1, Theorem III.3] and also included in [AT, Lemma 2.4]. For several reasons, we give here a third exposition of this proof. In contrast to [H1, III.2] and [AT, Lemma 2.3], we avoid the theory of polytopes and instead use some basic valuation theory and (inspired by [Bra, Lemma 1.10]) a fact from model theory. We believe that our viewpoint might be useful for the investigation of rings other than $A(g)$.

**Theorem 17** (Handelman). *For every $0 \neq g \in \mathbb{R}[\bar{X}]^+$ and $x \in S(T(g))$, there is some $w \in \mathbb{R}^n$ and $y \in (0,\infty)^n$ such that*

$$a(x) = \lambda_w(a)(y) \qquad \textit{for all } a \in A(g).$$

*Proof.* By Chevalley's Theorem [PD, A.1.10], we can extend the ring homomorphism $x : A \to \mathbb{R}$ to a place of $\mathbb{R}(\bar{X})$, i.e., we find a valuation ring $\mathcal{O} \supset A(g)$ of $\mathbb{R}(\bar{X})$ with maximal ideal $\mathfrak{m}$ and a ring homomorphism $\lambda : \mathcal{O} \to K$ into some extension field $K$ of $\mathbb{R}$ with kernel $\mathfrak{m}$ such that $\lambda|_{A(g)} = x$. Let $v : \mathbb{R}(\bar{X}) \to \Gamma \cup \{\infty\}$ be a valuation belonging to $\mathcal{O}$ where $\Gamma$ is (after extension) without loss of generality a nontrivial divisible ordered abelian group. Set

$$\Lambda := \{\alpha \in \mathrm{Log}(g) \mid v(\bar{X}^\alpha) = v(g)\} = \left\{\alpha \in \mathrm{Log}(g) \mid \lambda\left(\frac{\bar{X}^\alpha}{g}\right) \neq 0\right\}. \tag{47}$$

Now the first-order logic sentence

$$\exists u \exists v_1 \dots \exists v_n \left( \bigwedge_{\alpha\in\Lambda} \alpha_1 v_1 + \dots + \alpha_n v_n = u \wedge \bigwedge_{\alpha\in\mathrm{Log}(g)\setminus\Lambda} \alpha_1 v_1 + \dots + \alpha_n v_n > u \right)$$

in the language $\{+, <, 0\}$ holds in $\Gamma$ (take $v(g)$ for $u$ and $v(X_i)$ for $v_i$). It is a well-known fact in basic model theory that all nontrivial divisible ordered abelian groups satisfy exactly the same first-order sentences in this language [Mar, Corollary 3.1.17]. In particular, the above sentence holds in $\mathbb{R}$, i.e., we find $w \in \mathbb{R}^n$ and $c \in \mathbb{R}$

such that $\langle w, \alpha\rangle = c$ for all $\alpha \in \Lambda$ and $\langle w, \alpha\rangle > c$ for all $\alpha \in \operatorname{Log}(g) \setminus \Lambda$. It follows that $v_w(g) = -c$ and

$$\Lambda = \{\alpha \in \operatorname{Log}(g) \mid v_w(\bar{X}^\alpha) = v_w(g)\} = \left\{\alpha \in \operatorname{Log}(g) \mid \lambda_w\left(\frac{\bar{X}^\alpha}{g}\right) \neq 0\right\}. \tag{48}$$

In view of (47), (48) and (41), it remains only to show that there exists $y \in (0,\infty)^n$ such that

$$\lambda\left(\frac{\bar{X}^\alpha}{g}\right) = \lambda_w\left(\frac{\bar{X}^\alpha}{g}\right)(y) \qquad \text{for all } \alpha \in \Lambda. \tag{49}$$

Now set $m := \#\Lambda - 1 \in \mathbb{N}$ and write $\Lambda = \{\alpha^{(0)}, \dots, \alpha^{(m)}\}$. Assume for the moment that we have already shown the existence of some $y \in (0,\infty)^n$ satisfying

$$\lambda(\bar{X}^{\alpha^{(i)}-\alpha^{(0)}}) = y^{\alpha^{(i)}-\alpha^{(0)}} \qquad \text{for each } i \in \{1,\dots,m\}. \tag{50}$$

Then we get immediately that even

$$\lambda(\bar{X}^{\alpha^{(i)}-\alpha^{(j)}}) = y^{\alpha^{(i)}-\alpha^{(j)}} = \lambda_w(\bar{X}^{\alpha^{(i)}-\alpha^{(j)}})(y) \tag{51}$$

for $i, j \in \{0,\dots,m\}$. Writing $g = \sum_{\alpha\in\operatorname{Log}(g)} a_\alpha \bar{X}^\alpha$, we obtain

$$\begin{aligned}
\lambda_w\left(\frac{g}{\bar{X}^{\alpha^{(i)}}}\right)(y)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right) &= \sum_{\alpha\in\operatorname{Log}(g)} a_\alpha\lambda_w\left(\frac{\bar{X}^\alpha}{\bar{X}^{\alpha^{(i)}}}\right)(y)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right)\\
\overset{(48)}{=} \sum_{j=0}^m a_{\alpha^{(j)}}\lambda_w\left(\frac{\bar{X}^{\alpha^{(j)}}}{\bar{X}^{\alpha^{(i)}}}\right)(y)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right) &\overset{(51)}{=} \sum_{j=0}^m a_{\alpha^{(j)}}\lambda\left(\frac{\bar{X}^{\alpha^{(j)}}}{\bar{X}^{\alpha^{(i)}}}\right)\lambda\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right)\\
= \sum_{j=0}^m a_{\alpha^{(j)}}\lambda\left(\frac{\bar{X}^{\alpha^{(j)}}}{g}\right) &\overset{(47)}{=} \sum_{\alpha\in\operatorname{Log}(g)} a_\alpha\lambda\left(\frac{\bar{X}^\alpha}{g}\right) = \lambda\left(\frac{g}{g}\right) = \lambda(1) = 1
\end{aligned}$$

which shows (49). Therefore we are left with showing that there is some $y \in (0,\infty)^n$ fulfilling (50). Set $\beta^{(i)} := \alpha^{(i)} - \alpha^{(0)} \in \mathbb{Z}^n$ and $z_i := \lambda(\bar{X}^{\beta^{(i)}})$ for $i \in \{1,\dots,m\}$. Note that for all $i \in \{1,\dots,m\}$,

$$z_i = \underbrace{\lambda\underbrace{\left(\frac{\bar{X}^{\alpha^{(i)}}}{g}\right)}_{\in T}}_{\neq 0 \text{ by } (47)}\underbrace{\lambda\underbrace{\left(\frac{\bar{X}^{\alpha^{(0)}}}{g}\right)}_{\in T}}_{\neq 0 \text{ by } (47)}{}^{-1} > 0$$

since $\varphi(T) \subseteq [0,\infty)$. Using

$$y^{\beta^{(i)}} = y_1^{\beta_1^{(i)}} \cdots y_n^{\beta_n^{(i)}} = e^{(\log y_1)\beta_1^{(i)} + \dots + (\log y_n)\beta_n^{(i)}},$$

taking logarithms in (50) and rewriting it in matrix form, we therefore have to show that there are $y'_1, \dots, y'_n \in \mathbb{R}$ (corresponding to $\log y_1, \dots, \log y_n$) such that

$$\underbrace{\begin{pmatrix}\log z_1 & \dots & \log z_m\end{pmatrix}}_{=:L\in\mathbb{R}^{1\times m}} = \begin{pmatrix}y'_1 & \dots & y'_n\end{pmatrix}\underbrace{\begin{pmatrix}\beta_1^{(1)} & \dots & \beta_1^{(m)}\\ \vdots & & \vdots\\ \beta_n^{(1)} & \dots & \beta_n^{(m)}\end{pmatrix}}_{=:B\in\mathbb{R}^{n\times m}} \tag{52}$$

Provided now that $\ker B \subset \ker L$, the mapping $\operatorname{im} B \to \mathbb{R} : Bv \mapsto Lv$ ($v \in \mathbb{R}^m$) is a well-defined linear map and can be linearly extended to a map $\mathbb{R}^n \to \mathbb{R}$ represented by a $1 \times n$ matrix $\begin{pmatrix} y'_1 & \dots & y'_n \end{pmatrix}$ satisfying (52).

Finally, we show $\ker B \subset \ker L$. Since all entries of $B$ lie in the field $\mathbb{Q}$, $\ker B$ has a $\mathbb{Q}$-basis but then also $\mathbb{R}$-basis consisting of vectors $k \in \mathbb{Z}^m$. Therefore consider an arbitrary $k \in \mathbb{Z}^m$ with

$$\sum_{j=1}^{m} \beta_i^{(j)} k_j = 0 \qquad \text{for all } i \in \{1, \dots, m\}.$$

Taking the logarithm of

$$\begin{aligned} e^{(\log z_1)k_1 + \dots + (\log z_m)k_m} &= z_1^{k_1} \cdots z_m^{k_m} = \lambda(\bar{X}^{\beta^{(1)}})^{k_1} \cdots \lambda(\bar{X}^{\beta^{(m)}})^{k_m} \\ &= \lambda(\bar{X}^{\beta^{(1)}k_1 + \dots + \beta^{(m)}k_m}) = \lambda(\bar{X}^0) = \lambda(1) = 1 = e^0, \end{aligned}$$

we get indeed $k \in \ker L$. □

**Corollary 18** (Handelman)**.** *For every $0 \neq g \in \mathbb{R}[\bar{X}]^+$ and $x \in S(T(g))$, there exist $w \in \mathbb{R}^n$ and $y \in (0, \infty)^n$ such that*

$$a(x) = \lim_{t \to \infty} a(e^{tw_1} y_1, \dots, e^{tw_n} y_n) \qquad \text{for all } a \in A(g).$$

*Proof.* Rewrite the last theorem using (44) and (46). □

We need a little number theoretic fact to make Proposition 20 below available.

**Lemma 19.** *Suppose $l_1, l_2 \in \mathbb{N}$ are relatively prime in $\mathbb{Z}$. Then for all $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\mathbb{N} \cap [m, \infty) \subset (\mathbb{N} \cap [k, \infty))l_1 + (\mathbb{N} \cap [k, \infty))l_2$.*

*Proof.* Write $1 = s_1 l_1 + s_2 l_2$ with $s_1, s_2 \in \mathbb{Z}$. If $s_1, s_2 \geq 0$ then either $s_1 = l_1 = 1$ or $s_2 = l_2 = 1$. Given $k \in \mathbb{N}$, we then can set $r := k$. Hence suppose, say, $s_1 < 0$. Then necessarily $l_2, s_2 > 0$. Given $k \in \mathbb{N}$, set

$$r := (l_2 - 1)(-s_1)l_1 + kl_1 + kl_2 \in \mathbb{N}.$$

Now we have for all $i \in \mathbb{N}$ and $j \in \{0, \dots, l_2 - 1\}$,

$$r + il_2 + j = (k + (l_2 - 1 - j)(-s_1))l_1 + (k + js_2 + i)l_2.$$

□

**Proposition 20.** *Suppose $f \in \mathbb{R}[\bar{X}]$ and let $l_1, l_2 \in \mathbb{N}$ be relatively prime in $\mathbb{Z}$. If it is true for $f^{l_1}$ and $f^{l_2}$ that all its sufficiently high powers have nonnegative coefficients, then the same is true for $f$.*

**Lemma 21** (Handelman)**.** *Suppose $f \in \mathbb{R}[\bar{X}]$, $1 \leq l \in \mathbb{N}$ and $f^l \in \mathbb{R}[\bar{X}]^+$. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and for all vertices $\alpha$ (i.e., extreme points) of* $\operatorname{New}(f)$*,*

$$(lk - 1)\alpha + \operatorname{Log}(f) \subset \operatorname{Log}(f^{lk}).$$

*Proof.* It is convenient to work in the ring $\mathbb{R}[X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}] \subset \mathbb{R}(\bar{X})$ of Laurent polynomials. The Laurent monomials $\bar{X}^\alpha := X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ ($\alpha \in \mathbb{Z}^n$) form an $\mathbb{R}$-vector space basis of it. Extending the definitions in the obvious way, we can speak of $\operatorname{Log}(f) \subset \mathbb{Z}^n$ and $\operatorname{New}(f) \subset \mathbb{R}^n$ for any Laurent polynomial $f$. We now prove our claim even for Laurent polynomials $f$.

Since the polytope $\operatorname{New}(f)$ has only finitely many vertices, it suffices to show that the claimed inclusion of sets holds for a *fixed* vertex $\alpha$ and all large $k$. Replacing $f$

by $\bar{X}^{-\alpha}f$, we can assume right away that $\alpha = 0$. Because the origin is now a vertex of $\operatorname{New}(f)$, we can choose $w \in \mathbb{R}^n$ such that $\langle w, \beta\rangle > 0$ for all $0 \neq \beta \in \operatorname{Log}(f)$. For all $0 \neq \beta, \gamma, \delta \in \operatorname{Log}(f)$ with $\beta = \gamma + \delta$, in the equality $\langle w, \beta\rangle = \langle w, \gamma\rangle + \langle w, \delta\rangle$ the two terms on the right hand side are then smaller than the left hand side. We need the following consequence from this: Calling a nonzero element of $\operatorname{Log}(f)$ an *atom* if it is not a sum of two nonzero elements of $\operatorname{Log}(f)$, every element of $\operatorname{Log}(f)$ can be written as a finite sum of atoms (the origin being the sum of zero atoms). Since $\operatorname{Log}(f)$ is finite, we can choose $k \in \mathbb{N}$ such that every element of $\operatorname{Log}(f)$ is a sum of at most $k$ such atoms. On the other hand, because $f^l$ has nonnegative coefficients, $\operatorname{Log}(f^{lk})$ consists of the sums of $k$ elements of $\operatorname{Log}(f^l)$. Using $0 \in \operatorname{Log}(f)$, it is enough to show that all atoms are contained in $\operatorname{Log}(f^l)$. This is clear from the fact that an atom $\alpha$ can can be written as a sum of $l$ elements from $\operatorname{Log}(f)$ only in a trivial way. In fact, the coefficient of $\bar{X}^\alpha$ in $f^l$ is $l$ times the coefficient of $\bar{X}^\alpha$ in $f$ and therefore nonzero. $\square$

Now we are enough prepared to give a proof of Handelman's result based on our membership criterion.

**Theorem 22** (Handelman)**.** *Let $f \in \mathbb{R}[\bar{X}]$ be a polynomial such that $f^k$ has no negative coefficients for some $k \geq 1$ and $f(1,1,\dots,1) > 0$. Then for all sufficiently large $k \in \mathbb{N}$, $f^k$ has no negative coefficients.*

*Proof.* For any polynomial $p \in \mathbb{R}[\bar{X}]$, we write $p^+$ for the sum of its monomials with positive coefficients and $p^-$ for the negated sum of its monomials with negative coefficients. So we always have $p = p^+ - p^-$, $p^+, p^- \in \mathbb{R}[\bar{X}]^+$ and $\operatorname{Log}(p^+) \dot\cup \operatorname{Log}(p^-) = \operatorname{Log}(p)$. First, we prove the theorem under the additional assumption

$$\operatorname{in}_w(f) \in \mathbb{R}[\bar{X}]^+ \qquad \text{for all } w \in \mathbb{R}^n \text{ with } \operatorname{in}_w(f) \neq f. \tag{53}$$

By Lemma 21, we can choose $k \in \mathbb{N}$ such that $g := f^k$ has no negative coefficients and

$$(k-1)\alpha + \operatorname{Log}(f) \subset \operatorname{Log}(g) \qquad \text{for all vertices } \alpha \text{ of } \operatorname{New}(f). \tag{54}$$

Pick an arbitrary vertex $\alpha_0$ of $\operatorname{New}(f)$. Then we have for all $N \in \mathbb{N}$,

$$\begin{aligned} a := \frac{\bar{X}^{(k-1)\alpha_0} f}{g} &= \underbrace{\left(1 - N \overbrace{\sum_\alpha \frac{\bar{X}^{(k-1)\alpha} f^+}{g} \cdot \frac{\bar{X}^{(k-1)\alpha} f^-}{g}}^{=:c_1}\right)}_{=:b_1(N)} \underbrace{\frac{\bar{X}^{(k-1)\alpha_0} f^+}{g}}_{=:t_1} \\ &+ \underbrace{\left(N \overbrace{\sum_\alpha \frac{\bar{X}^{(k-1)\alpha} f^+}{g} \cdot \frac{\bar{X}^{(k-1)\alpha} f^+}{g}}^{=:c_2} - 1\right)}_{=:b_2(N)} \underbrace{\frac{\bar{X}^{(k-1)\alpha_0} f^-}{g}}_{=:t_2} \end{aligned} \tag{55}$$

where the indices of summation run over all vertices $\alpha$ of $\operatorname{New}(f)$. We will show that for $N$ sufficiently big, (55) serves as an identity like it is required in Theorem 10 which we are going to apply to the ring $A := A(g)$ together with its archimedean

semiring $T := T(g)$. To do this, first of all, observe that all fractions appearing in (55) lie in $A$ by (54).

Claim 1: $a > 0$ on $(0,\infty)^n$. From $f^k = g \in \mathbb{R}[\bar{X}]^+$, it follows that $a^k > 0$ on $(0,\infty)^n$. Using the continuity of $a$ on the connected space $(0,\infty)^n$, we obtain either $a > 0$ on $(0,\infty)^n$ or $a < 0$ on $(0,\infty)^n$. The latter can be excluded using the hypothesis $f(1,1,\dots,1) > 0$

Claim 2: $a \geq 0$ on $S(T)$. This follows from Claim 1 and Corollary 18.

Claim 3: $c_1 = 0$ on $S_{a=0}(T)$. Let $w \in \mathbb{R}^n$. According to Theorem 17, we would have to show that $\lambda_w(a)(y) = 0$ implies $\lambda_w(c_1)(y) = 0$ for all $y \in (0,\infty)^n$. In fact, we show that $\lambda_w(c_1) \neq 0$ implies $\lambda_w(a) = a$ which is clearly more by Claim 1. So suppose that $\lambda_w(c_1) \neq 0$. Then there is some vertex $\alpha$ of $\operatorname{New}(f)$ with $v_w(\bar{X}^{(k-1)\alpha} f^+) = v_w(g) = v_w(\bar{X}^{(k-1)\alpha} f^-)$. This implies $v_w(f^+) = v_w(f^-)$ whence $\operatorname{in}_w(f) \notin \mathbb{R}[\bar{X}]^+$. From (53), we now deduce $\operatorname{in}_w(f) = f$. This means that for all exponent tuples $\beta \in \mathbb{N}^n$ appearing in $f$, $\langle w, \beta \rangle = -v_w(f)$ is constant. Being vertices of $\operatorname{New}(f)$, both $\alpha_0$ and $\alpha$ are among these $\beta$. We obtain therefore $v_w(\bar{X}^{(k-1)\alpha_0} f) = (k-1)v_w(\bar{X}^{\alpha_0}) + v_w(f) = k v_w(f) = v_w(f^k) = v_w(g)$. Exploiting the definition (46) of $\lambda_w$ together with $\operatorname{in}_w(\bar{X}^{(k-1)\alpha_0} f) = \bar{X}^{(k-1)\alpha_0} \operatorname{in}_w(f) = \bar{X}^{(k-1)\alpha_0} f$ and $\operatorname{in}_w(g) = \operatorname{in}_w(f^k) = \operatorname{in}_w(f)^k = f^k = g$, we see that $\lambda_w(a) = a$.

Claim 4: $\operatorname{New}(f) = \operatorname{New}(f^+)$. Of course, we have $\operatorname{New}(f) \supset \operatorname{New}(f^+)$ since $\operatorname{Log}(f) \supset \operatorname{Log}(f^+)$. For the other inclusion, it clearly suffices to show that every vertex $\alpha$ of $\operatorname{New}(f)$, is contained in $\operatorname{Log}(f^+)$. But for such a vertex $\alpha$, $\operatorname{in}_w(f) = \{\lambda \bar{X}^\alpha\}$ for some $\lambda \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$. Except in the case where $f = \lambda \bar{X}^\alpha$, it follows from (53) that $\lambda > 0$ whence $\alpha \in \operatorname{Log}(f^+)$. If $f = \lambda \bar{X}^\alpha$, then $\lambda > 0$ follows from $f(1,1,\dots,1) > 0$.

Claim 5: $c_2 > 0$ on $S(T)$. Let $w \in \mathbb{R}^n$. By Theorem 17, $\lambda_w(c_2)(y) > 0$ for all $y \in (0,\infty)^n$ is what we would have to show. By definition of $\lambda_w$ it is enough to show that $\lambda_w(c_2) \neq 0$ since $\bar{X}^{(k-1)\alpha} f^+$ has no negative coefficients. We obtain from Claim 4 that $v_w(f^+) = v_w(f)$. Choose a vertex $\alpha$ of $\operatorname{New}(f)$ such that $v_w(f) = v_w(\bar{X}^\alpha)$. Then $v_w(\bar{X}^{(k-1)\alpha} f^+) = (k-1)v_w(\bar{X}^\alpha) + v_w(f^+) = k v_w(f) = v_w(f^k) = v_w(g)$. Therefore $\lambda_w(c_2) \neq 0$ as desired.

Regarded as a continuous real-valued function on the compact space $S(T)$, $c_2$ is bounded from below by some positive real number by Claim 5. Consequently, we can choose $N \in \mathbb{N}$ so large that $b_2(N) = Nc_2 - 1 > 0$ on the whole of $S(T)$, in particular on $S_{a=0}(T)$. By Claim 3, we have that $b_1(N) = 1 - Nc_1 = 1 > 0$ on $S_{a=0}(T)$. Of course, $t_1, t_2 \in T$. Altogether, we can apply Theorem 10 and see that $a \in T$. By definition of $T = T(g)$, this means that $g^m \bar{X}^{(k-1)\alpha_0} f \in \mathbb{R}[\bar{X}]^+$ for some $m \in \mathbb{N}$. Omitting $\bar{X}^{(k-1)\alpha_0}$ does not change this fact, so that $f^{km+1} = g^m f \in \mathbb{R}[\bar{X}]^+$. At the same time, of course, $f^{km} = g^m \in \mathbb{R}[\bar{X}]^+$. Proposition 20 yields now that all sufficiently high powers of $f$ lie in $\mathbb{R}[\bar{X}]^+$.

Thus we have shown the theorem under the assumption (53). Now in the general case, we proceed by induction on the number of monomials appearing in $f$. The case where $f$ has only one monomial is trivial. Now suppose that $f$ has at least two monomials. The hypothesis implies clearly that

$$f > 0 \qquad \text{on } (0,\infty)^n. \tag{56}$$

Let $w \in \mathbb{R}^n$ such that $\operatorname{in}_w(f)$ has less monomials than $f$. For some $k \geq 1$, $(\operatorname{in}_w(f))^k = \operatorname{in}_w(f^k) \in \mathbb{R}[\bar{X}]^+$ by the hypotheses on $f$. Evaluating this at

$(1,1,\dots,1)$, we see that $\mathrm{in}_w(f)$ does not vanish at this point. Moreover, it is nonnegative at the same point by (44) and (56). Altogether, we can apply the induction hypothesis on $\mathrm{in}_w(f)$ to get that $\mathrm{in}_w(f^k) = (\mathrm{in}_w(f))^k \in \mathbb{R}[\bar{X}]^+$ for all large $k$.

Since $\{\mathrm{in}_w(f) \mid w \in \mathbb{R}^n\}$ is of course finite, this shows that we find $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$ and $w \in \mathbb{R}^n$ with $\mathrm{in}_w(f) \neq f$, $\mathrm{in}_w(f^k) \in \mathbb{R}[\bar{X}]^+$. This shows that (53) is satisfied with $f$ replaced by $f^k$ for any $k \geq k_0$ (note that $\mathrm{in}_w(f^k) \neq f^k$ implies trivially $\mathrm{in}_w(f) \neq f$). In particular, we find $l_1, l_2 \in \mathbb{N}$ that are relatively prime in $\mathbb{Z}$ such that (53) holds with $f$ replaced by $f^{l_1}$ and $f^{l_2}$, e.g., take $l_1 := k_0$ and $l_2 := k_0 + 1$. By the special case of the theorem already proved, we get that $(f^{l_1})^k$ and $(f^{l_2})^k$ have no negative coefficients for all large $k$. According to Proposition 20, this means that all sufficiently high powers of $f$ have only nonnegative coefficients. $\square$

**Corollary 23** (Handelman). *If some odd power of a real polynomial in several variables has only nonnegative coefficients, then so do all sufficiently high powers.*

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